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ON THE GREEN'S FUNCTION FOR THE LINEARIZED VLASOV EQUATION

Harold Weitzner

15 January 1962

AEC Research and Development Report

NEW YORK UNIVERSITY

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Abstract

This paper considers the Vlasov equation without magnetic fields, linearized about a state with no electric fields or space and time variation. Integral representations for the Green's functions are presented for systems of one, two, and three dimensions. For the one-dimensional stable case, two asymptotic expansions of the electric field for large time are presented. Similar expansions are given for a specific example of a one-dimensional unstable plasma. In two and three dimensions, asymptotic expansions are exhibited for arbitrary plasmas having cylindrical or spherical symmetry.

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On the Green's Function for the Linearized Vlasov Equation

I. Introduction

One of the common models employed to study the nature of plasma oscillations consists of the linearized Vlasov equation without magnetic fields and the relevant equations from Maxwell's equations.(1) For the situation in which the unperturbed state about which linearization is performed is constant in space and time and has no electric field, we present integral representations of the Green's function for the equations.(2) In our treatment the electric field vector, velocity vector, and position vector all have the same number of components -- the dimension of the space. For "stable" plasmas we derive asymptotic expansions of the electric field for large time. In the one-dimensional case we also obtain asymptotic expansions for a specific unstable plasma and comment on the general unstable case. Following the treatment of Landau,1) we start by examining the Fourier-Laplace transforms of the equations. However, we proceed to study the solutions of the equations themselves rather than their transforms or the ensuing dispersion relation.

Characteristic of many problems of mathematical physics, the form of the solution of the equations depends on the dimension of the space. Also, we find it necessary to impose different conditions and to use varying techniques in one, two, or three dimensions. Unfortunately, this problem presents still other confusing aspects. First, a natural derivation of the integral representation of the Green's function does not seem feasible except in special cases. However, we prove that the proposed Green's functions do satisfy the equations. Secondly, in certain cases the solution is a distribution, or a weak solution, rather than an ordinary function. Thus, for simplicity we consider the relatively direct one dimensional case in some detail before proceeding to the more complicated two- and three-dimensional cases. In the next section we formulate the problem generally, and after a few comments we present some mathematical material applicable to all the various cases. The third section, devoted entirely to the one-dimensional case, contains a derivation of the solution of questionable rigor and a full proof that the solution obtained is in fact a solution to the problem. In the fourth section we derive two asymptotic expansions of the electric field for large time for an arbitrary, stable, one dimensional plasma. In the fifth section we consider a specific example of a one-dimensional plasma that is subject to the two stream instability and we obtain again asymptotic expansions of the electric field. We conclude that section with some brief remarks on unstable one-dimensional plasmas. In the sixth section we give the three-dimensional representations and then asymptotic expansions for the spherically symmetric case. The final section repeats much of the material of the sixth, but in two dimensions.

II. Formulation of the Problem

Under the hypotheses of the introduction we may assume that the particle distribution function, $f(\bar{r}, \bar{v}, t)$, and the electric field, $\bar{E}(\bar{r}, t)$, satisfy the equations:

$$\frac{\partial f}{\partial t}(\bar{r}, \bar{v}, t) + \bar{v}\cdot\frac{\partial f}{\partial \bar{r}} + \frac{e}{m}\bar{E}(\bar{r}, t)\cdot\frac{\partial f_o(\bar{v})}{\partial \bar{v}} = 0 \tag{1a}$$

$$\bar{\nabla}\cdot\bar{E}(\bar{r}, t) = 4\pi e \int f(\bar{r}, \bar{v}, t)(d\bar{v}) \tag{1b}$$

$$\bar{\nabla}\times\bar{E}(\bar{r}, t) = 0 \; . \tag{1c}$$

In the equations (1), $f_o(\bar{v})$ is the unperturbed distribution function and e and m are the charge and mass of the particles.

Without some statement concerning initial conditions and/or boundary conditions for the functions $\bar{E}(\bar{r}, t)$ and $f(\bar{r}, \bar{v}, t)$, we have not specified a complete problem. Since we do not prove any existence and uniqueness theorems for (1) with appropriate additional conditions, we can only propose what seem to be reasonable conditions. From the formulation of the problem we would certainly expect to give an initial condition for $f(\bar{r}, \bar{v}, t)$. Together (1b) and (1c) imply that $\bar{E}(\bar{r}, t)$ satisfies Poisson's equation, so that we might expect to impose only regularity conditions at infinity on $\bar{E}(\bar{r}, t)$. Thus we consider the Green's function for the system (1) to be functions satisfying (1) and the added conditions

$$f(\bar{r}, \bar{v}, 0) = g(\bar{v})\delta(\bar{r}) \tag{2a}$$

and

$$\lim_{\substack{|r| \to \infty \\ t \neq 0}} \bar{E}(\bar{r}, t) = 0 . \tag{2b}$$

Thus the Green's function represents the disturbance produced by a source of particles at $\bar{r} = 0$ and $t = 0$ with the velocity distribution $g(\bar{v})$. We may express the solution to the general initial value problem in terms of convolution integrals of $f(\bar{r}, \bar{v}, t)$ or $\bar{E}(\bar{r}, t)$ and the given initial data.

While we have set up the equations for a one-component plasma, the system of equations (1) covers the many-component case. For let us suppose that there were many species of particles, each with its own distribution function, $f_i(\bar{r}, \bar{v}, t)$, unperturbed distribution function $f_{oi}(\bar{v})$, charge e_i, and mass m_i, such that each distribution function satisfied (1a). The appropriate form for (1b) would be

$$\bar{\nabla} \cdot \bar{E}(\bar{r},t) = 4\pi \sum_{(i)} e_i \int f_i(\bar{r}, \bar{v}, t)(d\bar{v}). \tag{1b'}$$

If we define

$$f'(\bar{r}, \bar{v}, t) = \frac{1}{e} \sum_{(i)} e_i f_i(\bar{r}, \bar{v}, t).$$

and

$$f'_o(\overline{v}) = \frac{m}{e^2} \sum \frac{e_i^2}{m_i} \; f_{oi}(\overline{v}) \; ,$$

then $f'(\overline{r}, \overline{v}, t)$, $f'_o(\overline{v})$, and $\overline{E}(\overline{r}, t)$ again satisfy (1). Further, if $f_{oi}(\overline{v}) \geq 0$, then $f'_o(\overline{v}) \geq 0$, so that the problem is essentially the same as with only one species of particles.

It is well known and immediately verified that the system of equations (1) is invariant under the Gallilean transformation

$$\begin{aligned} \overline{u} &= \overline{v} - \overline{w} \\ \overline{r}' &= \overline{r} - \overline{w}t \\ t' &= t . \end{aligned}$$

Ultimately we expect to obtain asymptotic expansions of the electric field and we would certainly expect them to vary from one coordinate system to another. For reasons apparent later we choose one particular system as rest frame, the unique system in which

$$\int \overline{v} f_o(\overline{v})(d\overline{v}) = 0 \; . \tag{3}$$

In certain circumstances we must add an additional condition in order to obtain the Green's function. The restriction

$$\int g(\overline{v})(d\overline{v}) = 0 \; , \tag{4}$$

asserts that the perturbation does not change the amount of charge in the system. For example, in one dimension we derive from (1b) that $\frac{1}{4\pi}$ $(E(+\infty, t)-E(-\infty, t))$ equals the charge added by the perturbation. Thus if (2b) is to hold, then we must impose (4). In two dimensions we do not require (4) to hold. For technical reasons we start by imposing (4) in three dimensions, but we finally drop the condition.

Essentially all the rest of this paper considers each dimension separately. However, we present below certain suxilliary material which is independent of the dimension. Since we expect to employ Laplace transform techniques, we must study certain functions analytic in a half-plane. If we define the physical constants

(5a) $$n = \int f_o(\overline{v})d\overline{v}$$

and

(5b) $$\omega_p^2 = \frac{4\pi ne^2}{m} \,, \qquad \text{the plasma frequency,}$$

and introduce the function $\tilde{f}_o(\overline{v}) = f_o(\overline{v})/n$, then we define the functions

(6) $$G(z, \hat{k}) = -4\pi e \int \frac{g(\overline{v})(d\overline{v})}{\hat{k}\cdot\overline{v}-z} \,, \qquad \operatorname{Im} z > 0$$

and

(7) $$F(z, \hat{k}) = \omega_p^2 \int \frac{\hat{k}\cdot \frac{\partial\tilde{f}_o(\overline{v})}{\partial\overline{v}} (d\overline{v})}{\hat{k}\cdot\overline{v}-z} \,, \qquad \operatorname{Im} z > 0 \,.$$

where $\hat{k}$ is a real unit vector in the relevant space. In one dimension $\hat{k}$ is omitted from (6), (7) and the formulas below. We see immediately that $F(z, \hat{k})$ and $G(z, \hat{k})$ are analytic in the domain $\operatorname{Im} z > 0$ if $g(\bar{v})$ and $\partial f_o(\bar{v})/\partial\bar{v}$ are absolutely integrable. Further, if $g(\bar{v})$ and $\partial f_o(\bar{v})/\partial\bar{v}$ have derivatives which are continuous and absolutely integrable, then $F(z, \hat{k})$ and $G(z, \hat{k})$ may be extended continuously up to the line $\operatorname{Im} z = 0$. For many of the asymptotic expansions considered below we must assume that $\tilde{f}_o(\bar{v})$ and $g(\bar{v})$ have many continuous absolutely integrable derivatives, so that $F(z, \hat{k})$ and $G(z, \hat{k})$ may also possess many continuous derivatives in the domain $\operatorname{Im} z \geq 0$. However, since we never assume analyticity of $g(\bar{v})$ or $f_o(\bar{v})$, we cannot extend $F(z, \hat{k})$ or $G(z, \hat{k})$ analyticitally into the region $\operatorname{Im} z < 0$.

We also need asymptotic expansions of $F(z, \hat{k})$ and $G(z, \hat{k})$ near infinity in the half-plane $\operatorname{Im} z \geq 0$. Without full proof, from the identity

$$G(z, \hat{k}) = \frac{4\pi e}{z}\int g(\bar{v})(d\bar{v})\left\{1 + \frac{\bar{v}\cdot\hat{k}}{z} + \frac{(\bar{v}\cdot\hat{k})^2}{z^2} + \ldots + \frac{(v\cdot k)^n}{z^n} - \frac{(\bar{v}\cdot\hat{k})^{n+1}}{z^n(z-\bar{v}\cdot\hat{k})}\right\}$$

we see that if $g(\bar{v})$ and its derivative vanish as fast as $1/|\bar{v}|^n$ for n sufficiently large, then we may derive the asymptotic expansion

$$G(z, \hat{k}) = \frac{1}{z}\left\{\alpha_o + \frac{\alpha_1(\hat{k})}{z} + \ldots + \frac{\alpha_n(\hat{k})}{z^n} + \frac{A_n(z, \hat{k})}{z^n}\right\} \tag{8}$$

valid for Im $z \geq 0$. In (8), $|A_n(z, \hat{k})|$ is uniformly bounded for Im $z \geq 0$ and

$$(9) \qquad \alpha_n(\hat{k}) = 4\pi e \int (d\bar{v}) g(\bar{v}) (k \cdot \bar{v})^n .$$

When $g(\bar{v})$ is restricted so that (4) holds, we infer that $\alpha_o = 0$. In a similar manner we obtain the expansion for $F(z, \hat{k})$,

$$(10) \qquad F(z, \hat{k}) = \frac{\omega_p^2}{z^2} \left\{ 1 + \frac{\beta_2(\hat{k})}{z^2} + \frac{\beta_3(\hat{k})}{z^3} + \cdots + \frac{\beta_n(\hat{k})}{z^n} + \frac{B_n(z, \hat{k})}{z^n} \right\}$$

valid in Im $z \geq 0$ and with $|B_n(z, \hat{k})|$ uniformly bounded there. The coefficients $\beta_n(\hat{k})$ are defined by

$$(11) \qquad \beta_n(\hat{k}) = \int (\hat{k} \cdot \bar{v})^n \tilde{f}_o(\bar{v}) (d\bar{v}) .$$

We have used (3) to set $\beta_1(\hat{k}) = 0$ in (10).

III. The One-Dimensional Green's Function

In this section we present a derivation of the integral representations of the Green's function for the one dimensional case. The derivation is only appropriate for a "stable" plasma, but we show that the Green's function obtained is correct for any plasma. We noted above that we assume (4) holds so that in (8) $\alpha_o = 0$. In this case we also ignore equation (1c) which is trivially satisfied.

If we perform a Fourier transform in x and a Laplace transform in t, we obtain from equations (1a), (1b) and (2a) the relations

$$(12a) \qquad (p + ikv)f(k,v,p) + \frac{e}{m} E(k,p) \frac{\partial f_o(v)}{\partial v} = g(v), \quad \operatorname{Re} p > 0$$

and

$$(12b) \qquad ik\, E(k,p) = 4\pi e \int f(k,v,p)dv \; , \quad \operatorname{Re} p > 0.$$

We may solve (12a) for $f(k,v,p)$ in terms of $E(k,p)$ and then from (12b) we get, as in Landau's paper,

$$(13) \qquad E(k,p) = \frac{G(ip/k)}{k^2 - F(ip/k)} \quad , \qquad k > 0, \quad \operatorname{Re} p > 0,$$

where $G(z)$ and $F(z)$ are defined by (6) and (7). From (13) we recognize the well-known dispersion relation to be

$$(14) \qquad k^2 - F(ip/k) = 0 \; , \qquad k > 0, \quad \operatorname{Re} p > 0.$$

According to the dispersion relation the plasma is stable if $F(z)$ is never real and non-negative for $\operatorname{Im} z \geq 0$. Returning to the function $E(x,t)$, we derive

$$(15) \qquad E(x,t) \quad \frac{\operatorname{Re}}{2\pi^2} \frac{1}{i} \int_0^\infty dk \int_{\sigma - i\infty}^{\sigma + i\infty} dp \; e^{pt} \; e^{ikx} \frac{G(ip/k)}{k^2 - F(ip/k)} ,$$

where $\sigma > 0$ and all roots of (14) have $\operatorname{Re} p < \sigma$. In obtaining (15) we have used the representation for the delta function

$$\delta(x) = \frac{\operatorname{Re}}{\pi} \int_0^\infty e^{ikx} \, dk .$$

If we assume the plasma is stable, then we may displace the contour of the p integral in (15) to the left until $\sigma = 0$. We may now define the complex variable $z = ip/k$, and (15) becomes

$$(16) \qquad E(x,t) = \frac{\operatorname{Re}}{2\pi^2} \int_0^\infty k\,dk \int_{-\infty}^\infty dz \; e^{ik(x-zt)} \frac{G(z)}{k^2 - F(z)} .$$

Let us consider

$$(17) \qquad I = \int_{-\infty}^0 dz \; e^{ik(x-zt)} \frac{G(z)}{k^2 - F(z)} , \qquad k < 0 .$$

Since the denominator of the integrand in (17) never vanishes for $\operatorname{Im} z \geq 0$, we may displace the contour of integration in the z plane to any line $\operatorname{Im} z = \sigma' > 0$. As $\sigma' \rightarrow \infty$, the integrand vanishes and so we infer $I = 0$. We may then extend the k integration in (16) to minus infinity so that

$$E(x,t) = \frac{\mathrm{Re}}{2\pi^2} \int_{-\infty}^{\infty} k\,dk \int_{-\infty}^{\infty} dz\, \frac{G(z)}{k^2 - F(z)}\, e^{ik(x-zt)} .$$

The k integration is now trivial and we derive

$$(18) \qquad E(x,t) = \frac{\mathrm{Im}}{\pi} \int_{x/t}^{\infty} dz\, G(z) \cos(\sqrt{F(z)}\,(x-zt)) .$$

We may get f(x,v,t) most easily from (1a), using (2a) and (18), and we find

$$(19) \qquad f(x,v,t) = -\frac{e}{m\pi}\, \mathrm{Im}\, \frac{\partial f_o(v)}{\partial v} \int_{x/t}^{\infty} dz\, G(z)\, \frac{\sin(\sqrt{F(z)}(x-zt))}{\sqrt{F(z)}\,(v-z)}$$

$$+ g(v)\delta(x-vt).$$

We obtained (18) and (19) only under the severe assumption of stability of the plasma and we omitted numerous troublesome technicalities. Now we show generally that (18) and (19) are the Green's function. First we must show that (18) and (19) are well-defined, next that they satisfy the boundary condition and initial condition, and finally that they obey the differential equations. We know $F(z)$ is analytic in the domain $\mathrm{Im}\ z \geq 0$, but $\sqrt{F(z)}$ might not be single valued in the domain or analytic at all interior points. Since the integrands of (18) and (19) are even functions of $\sqrt{F(z)}$ we see that in their dependence on $F(z)$ they are analytic functions of z in $\operatorname{Im} z \geq 0$. From the asymptotic expansion for $F(z)$ we see that for fixed x and t the arguments of the sine and cosine in (18) and (19) are uniformly bounded in the domain. From the asymptotic expansion for $G(z)$, (with $\alpha_o = 0$), we infer that the integrals in (18) and (19) converge absolutely and uniformly near infinity. Further, the contours of integration in (18) and (19) may be taken from $\frac{x}{t}$ to any point at infinity in the half plane $\operatorname{Im} z \geq 0$. We have just shown that $E(x,t)$ is well-defined, but to show that $f(x,v,t)$ is well-defined we must show one more thing: that the singularity in the integrand of (19) at $z = v$ has no effect. For $\frac{x}{t} \neq v$ we may choose the path of integration so that it does not pass through $z = v$ and for $\frac{x}{t} = v$ there is no pole of the integrand since the numerator and denominator both have a first order zero at $z = v$. Thus $f(x,v,t)$ is well-defined and with little extra effort we might even show that $(f(x,v,t) - g(v)\delta(x-vt))$ is a continuous function of its arguments.

We verify first the easier of the initial and boundary conditions; corresponding to (2a) we show that

$$(20) \qquad \lim_{t \to 0} (f(x,v,t) - g(v)\delta(x-vt)) = 0$$

If $x \neq 0$ then (20) is trivial as the domain of integration in (19) becomes smaller and smaller as $t \to 0$; if $x = 0$, then the integrand tends to zero uniformly as $t \to 0$ and again (20) follows. To show that (2b) holds is more complicated. We choose a path of integration outside the circle $|z| = |x/t|$. From the asymptotic expansion for $F(z)$ in the form $F(z) = A(z)/z^2$, with $A(z)$ uniformly bounded, we see that on the path of integration $|\sqrt{F(z)}\, x| \leq ct$, for some c independent of x and t. Thus it follows that

$$|E(x,t)| \leq c'' \int_{x/t}^{\infty} |G(z)||dz|\, e^{c't} ,$$

where c' and c'' are constants independent of x and t. From the asymptotic expansion for $G(z)$, we obtain an acceptable form for (2b)

$$(21) \qquad |E(x,t) \leq \frac{c''te^{c't}}{|x|} .$$

It remains to show that the differential equations are satisfied, and again we start with the easier, (1b). From (18) and (19) we derive immediately that

$$(22) \qquad \frac{\partial E(x,t)}{\partial x} - 4\pi e \int f(x,v,t)dv = -\frac{1}{t\pi} \operatorname{Im} G(\tfrac{x}{t}) - \frac{4\pi e}{t} g(\tfrac{x}{t}) .$$

But from the integral representation for $G(z)$ and well-known formulas we infer that $\operatorname{Im} G(x/t) = -4\pi^2 e\, g(x/t)$. Thus the right hand side of (22) vanishes and (1b) holds.

If we were to try to verify directly the remaining equation, (1a), we would find that f(x,v,t) as given by (19) is not differentiable with respect to x and t in the whole x,v,t space. Since we have obtained a Green's function we might expect it to be a distribution, which need not be differentiable in the ordinary sense. Conveniently we may show (1a) holds without any of the complications of distribution theory. If we define new variables x' and t' by

$$x' = x - vt \tag{23a}$$

$$t' = t, \tag{23b}$$

then (1a) becomes, written in terms of both the old and new variables,

$$\frac{\partial f(x,v,t)}{\partial t'} + \frac{e}{m} E(x,t) \frac{\partial f_o(v)}{\partial v} = 0 . \tag{24}$$

If we define f'(x',v',t') = f(x,v,t), then (19) becomes

$$\begin{aligned} f'(x',v',t') &= -\frac{e}{m\pi} \operatorname{Im} \frac{\partial f_o(v)}{\partial v} \int_{\frac{x'}{t'}+v}^{\infty} dz\, \frac{G(z)}{v-z} \frac{\sin(\sqrt{F(z)}(x'-(z-v)t'))}{\sqrt{F(z)}} \\ &\quad + g(v)\delta(x') . \end{aligned} \tag{25}$$

We may now readily verify that f'(x',v',t') is continuously differentiable with respect to t' and that (24) holds. In a weakened sense we have thus shown that (1a) is satisfied. Hence (18) and (19) are the solutions to the system of equations (1) and (2), (with (4) assumed).

IV. Asymptotic Expansions of the Electric Field in the One-Dimensional, Stable Case.

Although we now have an integral representation for the electric field in one dimension, we cannot learn very much from the formula as it is. Hence, in this and the succeeding section we obtain asymptotic expansions of the formula (18). Here we restrict ourselves to stable plasmas, for which, according to the preceeding discussion, F(z) is never real and non-negative. We consider two separate limits, x fixed and $t \to \infty$, and $x/t = v$ fixed and $t \to \infty$. The meaning of the first limit is clear; in one interpretation of the second limit we move out on a possible wave front traveling with speed v. Alternately, after a Gallilean transformation into a coordinate system moving with velocity v with respect to the rest frame, the limit x' fixed, $t' \to \infty$ is just the same as the second one above. Because ordinary saddle-point methods for obtaining asymptotic expansions are not valid for a stable plasma, we must employ the involved procedures given below. We consider first the easier limit, in which $x/t = v$ is held fixed as $t \to \infty$.

Since $F(z)$ is never real and non-negative we may define $\sqrt{F(z)}$ as a single valued analytic function in the half-plane $\operatorname{Im} z \geq 0$ and we may also specify that $\operatorname{Im} \sqrt{F(z)} < 0$ there. If we set

$$(26) \qquad \phi(z) = \phi(z,v) = (v-z)\sqrt{F(z)} \ ,$$

then on the line $\operatorname{Im} z = 0$

$$(27a) \qquad \operatorname{Im} \phi(z,v) > 0 \quad \text{if} \quad z > v$$

and

$$(27b) \qquad \operatorname{Im} \phi(z,v) < 0 \quad \text{if} \quad z < v.$$

For simplicity we temporarily assume that $\frac{d\phi(z,v)}{dz}$ never vanishes on $\operatorname{Im} z = 0$. We remove this restriction below.

After these preliminaries we may now easily obtain the desired expansion by repeated integrations by parts. On rewriting (18), we have

$$(28) \qquad E(x,t) = E(vt,t) = \frac{\operatorname{Im}}{2\pi}\left\{\int_v^{\infty} dz\ G(z)e^{it\phi(z,v)} + \int_v^{-\infty} dz\ G(z)e^{-it\phi(z,v)}\right\}$$

After the first integration by parts we derive

$$(29)\qquad E(vt,t) = \frac{\mathrm{Im}}{2\pi}\left\{\frac{G(z)}{it\phi'(z)}\, e^{it\phi(z)}\,\Bigg|_{v}^{\infty} + \frac{G(z)e^{-it\phi(z)}}{-it\phi'(z)}\,\Bigg|_{v}^{-\infty}\right\}$$

$$-\frac{\mathrm{Im}}{2\pi}\left\{\frac{1}{it}\int_{v}^{\infty} dz\; e^{it\phi(z)}\left(\frac{G(z)}{\phi'(z)}\right)' - \frac{1}{it}\int_{v}^{-\infty} dz\; e^{-it\phi(z)}\left(\frac{G(z)}{\phi'(z)}\right)'\right\}$$

We have obtained (29) under the tacit assumption that the two infinite integrals converge. From the asymptotic expansions of $F(z)$ and $G(z)$ we see that

$$\frac{G(z)}{\phi'(z)} \sim \frac{\alpha_1 + \alpha_2/z + \dots}{(v + \beta_2/z)(-\omega_p)} \quad .$$

Provided $v \neq 0$ we obtain $(G(z)/\phi'(z))' \sim c/z^2$ so that the integrands in (29) behave just like those in (28). Moreover we infer from (27) that the two exponentials in (28) or (29) are uniformly bounded in absolute value independently of t. The properties just described hold after any number of integrations by parts, so that after three more times we get

$$(30)\qquad E(vt,t) = -\frac{\mathrm{Im}}{\pi t^2}\left\{\left(\frac{G(z)}{\frac{d\phi}{dz}(z,v)}\right)' \frac{1}{\frac{d}{dz}\phi(z,v)}\,\Bigg|_{z=v}\right\} + O\left(\frac{1}{t^4}\right) .$$

We may obtain as many terms in the expansion as we please provided $f_o(v)$ and $g(v)$ are sufficiently differentiable. We note that more than the moments of $f_o(v)$ and $g(v)$ appear in (30).

In order to relax the restriction that $\frac{d\phi}{dz}(z,v) \neq 0$ on $\mathrm{Im}\ z = 0$, we remark first that if $d\phi/dz$ vanishes at $z = v$ then $F(z) = 0$, contrary to the assumption that F is never real and non-negative. Since $d\phi/dz \sim - \omega_p v/z^2$ we see that near infinity $d\phi/dz \neq 0$ on $\mathrm{Im}\ z = 0$. Thus we may enclose the zeros of $d\phi/dz$ in intervals bounded away from $z = v$ and z infinite. We may divide the integrals in (28) into integrals over the special intervals and the remainder of the line. On the special intervals the absolute value of the integrand vanishes at least as fast as $e^{-\alpha t}$ as $t \to \infty$ for some fixed $\alpha > 0$. Thus these terms vanish exponentially as $t \to \infty$. On the remainder of the line, integration by parts reproduces the same series (30).

The asymptotic expansion for x fixed and $t \to \infty$ is much more complicated. In some sense, infinity is a stationary point of the term in the exponent $z\sqrt{F(z)}$ and we must include it properly. To study this limit we must consider the function $\psi(z) = z\sqrt{F(z)}$, analytic in the half-plane $\text{Im } z \geq 0$. From the asymptotic expansion of $F(z)$, we obtain

$$(31) \qquad \psi(z) \sim \omega_p\left(1 + \frac{\beta_2}{2z^2} + \frac{\beta_3}{2z^3} + \ldots\right) .$$

Since the first two terms of the asymptotic expansion of $\psi(z)$ are real for z large and on the positive imaginary axis, we infer that there is a curve on which $\psi(z)$ is real near the positime imaginary axis near infinity. This curve cannot have both ends at infinity, for then the harmonic function $\text{Im } \psi(z) \sim \text{Im}(\beta_2\omega_p/2z^2)$ would vanish on the curve and at infinity, so that $\text{Im } \psi(z) \equiv 0$, an impossibility. The curve can have its end not at infinity only on the real axis, and only $z = 0$ is acceptable on that axis. By similar arguments there can be no other curves on which $\text{Im } \psi(z) = 0$ and this curve cannot cross itself. We sketch the situation in Figure 1 where we also define two contours, ℓ_1 and ℓ_2.

We start from essentially the same formula as (28):

$$(32) \qquad E(x,t) = \frac{\text{Im}}{2\pi}\left\{ \int_{\substack{x/t \\ \text{on } \ell_1}}^{\infty} dz\; G(z)e^{i\sqrt{F(z)}(x-zt)} + \int_{\substack{x/t \\ \text{on } \ell_2}}^{\infty} dz\; G(z)e^{-i\sqrt{F(z)}(x-zt)} \right\}$$

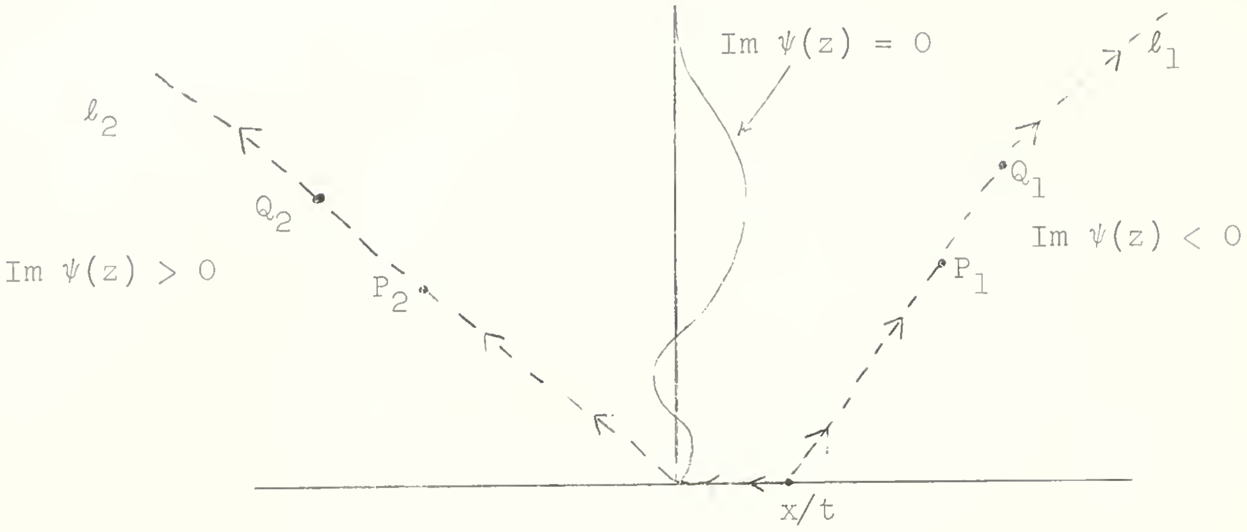


Figure 1

Curve Im $\psi(z) = 0$ and definition of the contours ℓ_1 and ℓ_2

On ℓ_1, Im $z\sqrt{F(z)}$ is negative so that the exponential factor in the first integral is a decreasing function of t. Similarly from zero to infinity on ℓ_2 the second integrand is a decreasing function of t while from zero to x/t on ℓ_2 the real part of the exponent of the integrand is negative. Thus any error terms in the asymptotic series, which are of the form of (32) are all uniformly bounded independently of t.

We use a simple formal device to separate the contributions to (32) from the neighborhood of infinity and from the neighborhood of $z = x/t$. We define an infinitely differentiable function of one variable on the path $\ell_1 - \ell_2$, $\widetilde{G}(z)$, such that $\widetilde{G}(z)$ equals the first n terms of the asymptotic expansion of $G(z)$ when z is on ℓ_1 from Q_1 to infinity or on ℓ_2 from

Q_2 to infinity. Further we assume that $\tilde{G}(z)$ vanishes identically from P_1 to x/t to zero to P_2. We may select Q_1 and Q_2 as far out as we please and n as large as we please, but we may not let n be infinite or Q_1 or Q_2 be at infinity. If we set $\mathcal{F}(z) = G(z) - \tilde{G}(z)$, then on $\ell_1 - \ell_2$, where $\mathcal{F}(z)$ is defined, $\mathcal{F}(z) = O(1/z^{n+1})$ and also $\mathcal{F}(z) \equiv G(z)$ near $z = x/t$.

We may now attain the separation of contributions from $z = x/t$ and infinity by writing

$$E(x,t) = E^{(1)}(x,t) + E^{(2)}(x,t) \tag{33a}$$

where

$$E^{(1)}(x,t) = \frac{\text{Im}}{2\pi}\left\{\int_{\substack{x/t\\ \text{on } \ell_1}}^{\infty} dz\ \mathcal{F}(z)e^{i\sqrt{F(z)}(x-zt)} + \int_{\substack{x/t\\ \text{on } \ell_2}}^{\infty} dz\ \mathcal{F}(z)e^{-i\sqrt{F(z)}(x-zt)}\right\} \tag{33b}$$

and

$$E^{(2)}(x,t) = \frac{\text{Im}}{2\pi}\left\{\int_{\substack{x/t\\ \text{on } \ell_1}}^{\infty} dz\ \tilde{G}(z)e^{i\sqrt{F(z)}(x-zt)} + \int_{\substack{x/t\\ \text{on } \ell_2}}^{\infty} dz\ \tilde{G}(z)e^{-i\sqrt{F(z)}(x-zt)}\right\} \tag{33c}$$

Since $\mathcal{F}(z)$ is $O(1/z^{n+1})$ we may integrate by parts several times in equation (33b) and we obtain

$$E^{(1)}(x,t) = -\frac{\mathrm{Im}}{\pi t^2}\left\{\left(\frac{G(z)}{\psi'(z)}\right)^1 \frac{1}{\psi'(z)}\Bigg|_{z=x/t}\right\} + O(1/t^3).$$

Omitting contributions vanishing exponentially we may take the first (second) integral in (33c) from Q_1 (Q_2) to infinity on ℓ_1 (ℓ_2). On the resulting contours of integration in (33c) we have chosen $\tilde{G}(z)$ to be the first n terms of the asymptotic expansion of $G(z)$. Without giving a full proof we wish to show that we may now insert the asymptotic expansion for $\psi(z)$ in the exponent. In essence the procedure is justified because the two leading terms in the expansion of $i\psi(z)$ produce a decreasing exponential the real part of which (for Q large enough) is larger in magnitude than the error term. On introducing the first few terms of the expansion of $\psi(z)$ and $\sqrt{F(z)}$, we obtain

$$(34a)\qquad E^{(2)}(x,t) = \frac{\mathrm{Im}}{2\pi}\left\{\int_{\substack{Q_1\\ \text{on } \ell_1}}^{\infty} dz\, \chi(z,x,t) + \int_{\substack{Q_2\\ \text{on } \ell_2}}^{\infty} dz\, \chi(z,-x,-t)\right\}$$

where

$$(34b)\qquad \chi(z,x,t) = e^{-i\omega_p t(1+\beta_2/2x^2)}\left(\frac{\alpha_1}{z^2} + \frac{\alpha_2}{z^3} + \frac{\alpha_3}{z^4} + \dots\right)\times$$

$$\left\{1 + \frac{ix\omega_p}{z} - \frac{x^2\omega_p^2}{z^2} + \frac{i\omega_p}{2z^3}(\beta_2 x - \beta_3 t) - \frac{ix^3\omega_p^3}{6z^3} + \right.$$

$$+ \frac{i\omega_p}{2z^4} (\beta_3 x - t(\beta_4 - \beta_2^2/4)) - \frac{x\omega_p^2}{2x^4} (\beta_2 x - \beta_3 t)$$

$$+ \frac{i\omega_p}{2z^5} \left\{(\beta_4 - \beta_2^2/4)x - (\beta_5 - \frac{\beta_2\beta_3}{2})t\right\} - \frac{x\omega_p^2}{2z^5} (\beta_3 x - t(\beta_4 - \beta_2^2/4))$$

$$\left. - \frac{ix^2\omega_p^2}{4z^5} (\beta_2 x - \beta_3 t) - \frac{\omega_p^2 \beta_3^2 t^2}{8z^6} + \ldots \right\} .$$

We may now extend the integrals to zero by adding parts which vanish exponentially. The integrals are now direct and we obtain:

$$E^{(2)}(x,t) = \frac{1}{\sqrt{\pi\omega_p\beta_2 t}} \left\{ \alpha_1 \cos(\omega_p t + 3\pi/4) \right.$$
$$+ \frac{\cos(\omega_p t + \pi/4)}{\omega_p\beta_2 t} \left(\frac{3}{2} (\frac{\beta_4}{\beta_2} - \frac{\beta_2}{4}) \alpha_1 + \frac{x^2\omega_p^2\alpha_1}{2} \right.$$
$$\left. + \frac{3}{2} \beta_3\alpha_2 - \alpha_3 \right) + \frac{\sin(\omega_p t + \pi/4)}{\omega_p\beta_2 t} \left(\frac{3x\omega_p\beta_3\alpha_1}{2\beta_2} \right.$$
$$\left.\left. - x\omega_p\alpha_2 \right) \right\} + O(1/t^{5/2}) . \tag{35}$$

In $E^{(2)}(x,t)$ we see the net effect of Landau damping in the reciprocal power decay in time. The long range and collective time nature of plasma oscillations is apparent in the x independence of the leading term of the expansion.

V. Asymptotic Expansions of the Electric Field for a One-Dimensional System Exhibiting the Two-Stream Instability.

Continuing the investigations of the previous section we now examine a specific example of an unstable plasma and obtain asymptotic expansions for the electric field. We consider the special case of a plasma that exhibits the two stream instability. An unperturbed distribution function of the form

$$(36) \qquad \tilde{f}_o(v) = \frac{2}{\sqrt{\pi}} v^2 e^{-v^2}$$

describes such a plasma. The largest part of this section is concerned with the properties of the analytic functions $F(z)$, $z^2F(z)$, $(v-z)^2F(z)$, their derivatives and their square roots. After such a study the asymptotic expansions are readily derived. In examining the functions, the questions we seek to answer are: (1) Does the exponent of the integrand in the integral for $E(x,t)$ have a saddle point? (2) If there is a saddle point, is it correct to use ordinary saddle point evaluation of the integral? (3) If there is no saddle point or if the ordinary evaluation of saddle point integrals fails, how is the integral approximated? We find that there is always a saddle point but in certain cases the saddle point contribution is negligible compared with the end point terms.

From the definition of $F(z)$ and (36) we obtain easily that

$$(37) \qquad F(z) = - \frac{4\omega_p^2}{\sqrt{\pi}} \left\{ \sqrt{\pi}\,(z^2 - \tfrac{1}{2}) + (z^3 - z) \int_{-\infty}^{\infty} \frac{e^{-u^2}}{u-z}\, du \right\} .$$

If we set $z = x + iy$, then

$$(38) \qquad \int \frac{e^{-u^2}}{u-z}\,du = \int \frac{u - x + iy}{(u-x)^2 + y^2}\, e^{-u^2} du \ .$$

From (37) we see that on the real axis Im F(z) = 0 only at $z = \pm 1$ and at zero, and at these points Re F(z) is negative and positive respectively. Equations (38) and (37) imply that F(z) is real on the imaginary axis. Since F(z) is real and positive at z = 0 and real and negative on the imaginary axis near infinity, it follows that at some point z_o on the imaginary axis $F(z_o) = 0$. Further, since $F(z_o) = 0$ and at infinity F(z) = 0, Rolle's theorem implies that $F'(z_1) = 0$ for some z_1 on the imaginary axis between z_o and infinity. We use the principle of the argument to count the zeros of F(z) in the upper half plane.

As the variable z follows the contour of Figure 2a, the argument of F(z), shown in Figure 2b, changes by 2π. Hence,

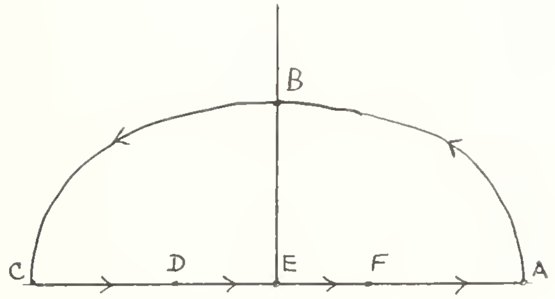


Figure 2a
Definition of the contour used to count the zeros of F(z)

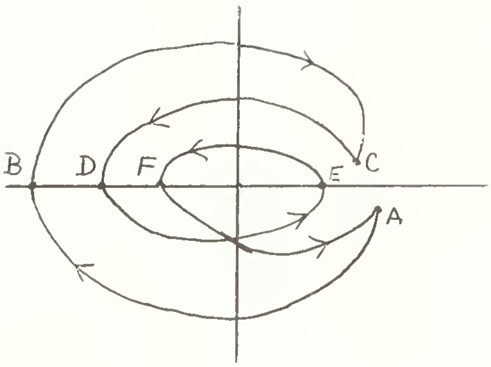


Figure 2b
Path of the argument of F(z) as z follows the contour of Fig.2a

we infer that F(z) has one zero in the upper half plane and it must be at z_o.

From the equation for F(z) we derive immediately that

$$F'(z) = -\frac{4\omega_p^2}{\sqrt{\pi}}\left\{-2\sqrt{\pi}\,(z^3-z)+(-1+5z^2-2z^4)\int_{-\infty}^{\infty}\frac{e^{-u^2}}{u-z}\,du\right\}. \tag{39}$$

On the real axis Im F'(z) vanishes at the four real roots of $2z^4 - 5z^2 + 1 = 0$. For z real and positive Re F'(z) is positive at the larger root of the polynomial and negative at the smaller root. For z real and negative the same relations hold. To count the zeros of F'(z) we use Figures 3a and 3b.

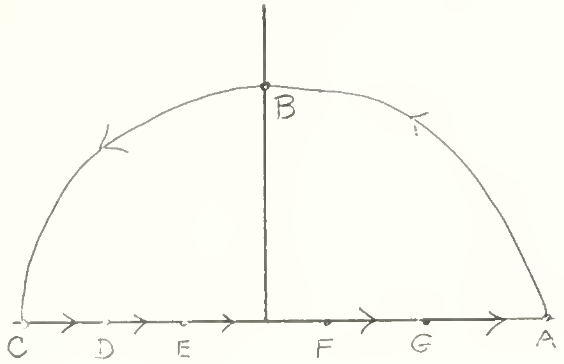


Figure 3a
Definition of the contours used to count the zeros of F'(z)

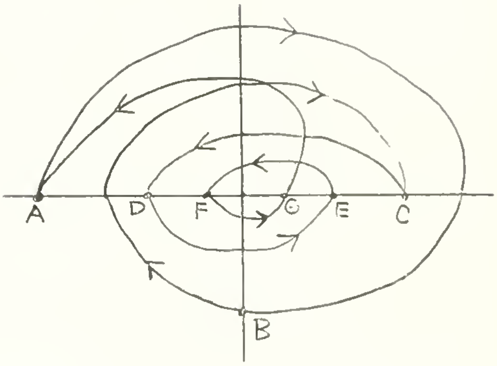


Figure 3b
Path of the argument of F'(z) as z follows the contour of Fig.3a

Thus F'(z) has one zero in the upper half plane and z_1 is that zero. It is useful to examine whether Im $\frac{F(z)}{F'(z)}$ ever vanishes on the real axis. If Im $\frac{F(z)}{F'(z)} = 0$ then Re F(z) Im F'(z) - Re F'(z) Im F(z) = 0. From equations (37) and (39) we infer that

$\text{Im}\,\frac{F(z)}{F'(z)} = 0$ on the real axis only if $z^2 + 1 = 0$. Hence, $\text{Im}\,\frac{F(z)}{F'(z)}$ is of one sign there.

We are now prepared to draw several contour plots of F(z) and related functions. In Figure 4 we plot the curves on which Im F(z) = 0.

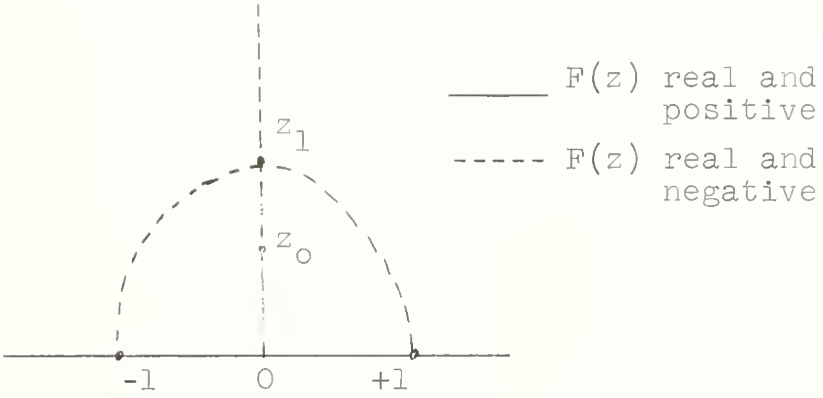


Figure 4
Curves Im F(z) = 0

We know that F(z) is real on the imaginary axis and it is positive or negative there as shown in Figure 4. Since $F'(z_1) = 0$, four lines leave z_1 on which F(z) is real. By symmetry, the two lines not yet accounted for either both go to the real axis or to infinity. Since Im F(z) is harmonic and vanishes at infinity, the two lines cannot go to infinity of $\text{Im}\,F(z) \equiv 0$. Thus the lines leaving z_1 connect as shown in Figure 4. By similar arguments there are no other curves on which Im F(z) = 0.

Next we consider the function $\psi^2(z) = z^2 F(z)$. On the imaginary axis $\psi^2(z)$ is real and $\psi^2(0) = \psi^2(z_o) = 0$. For some z_2 on the imaginary axis between z_o and zero $\frac{d}{dz}\,\psi^2(z) = 0$.

Just as for F(z) we obtain the contour plot of Figure 5.

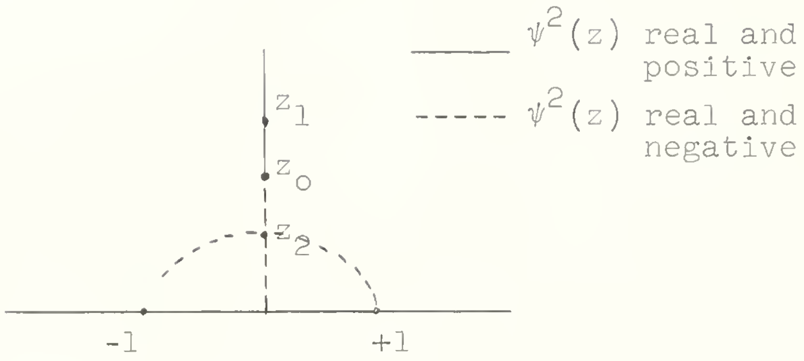


Figure 5
Curves Im $\psi^2(z) = 0$

In treating the stable plasma we introduced the function $\phi(z,v)$ by a relation equivalent to

$$\phi^2(z,v) = (v-z)^2 F(z) \ . \tag{40}$$

We are again interested in this function, and in order to use saddle point methods we must find values of z for which $\frac{d}{dz}\phi^2(z,v) = 0$. Such values satisfy the equation

$$v = \frac{2F(z)}{F'(z)} + z \ .$$

We are naturally lead to introduce the function

$$\chi(z) = z + \frac{2F(z)}{F'(z)} = \frac{1}{zF'(z)} \frac{d}{dz} \psi^2(z) \tag{41}$$

such that the real values of $\chi(z)$ correspond to points z' and values of $v = \chi(z')$ such that $\frac{d}{dz}\phi^2(z',v) = 0$. On the imaginary axis, where $\chi(z)$ is defined it is imaginary. Thus curves on which $\operatorname{Im}\chi(z) = 0$ can cross the imaginary axis only at z_1 (where $\chi(z)$ has a simple pole) or z_2 (where $\chi(z)$ has a simple zero). As $\chi(z)$ is not real on the real axis, we may draw the curves on which $\operatorname{Im}\chi(z) = 0$ as in Figure 6. On the

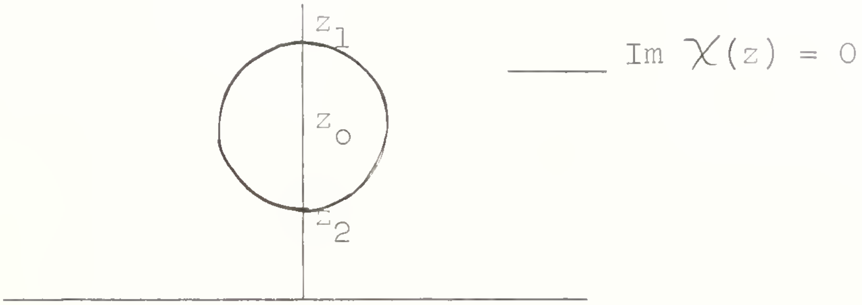


Figure 6
Curves $\operatorname{Im}\chi(z) = 0$

contour on which $\operatorname{Im}\chi(z) = 0$, $\chi(z)$ assumes all real values at least once. We have not really shown that the curves leaving z_1 and z_2 do not meet at infinity. Using the asymptotic expansion for $\chi(z)$ we might actually show this is not the case, but as we do not need this precise statement we pass over the question.

The function we wish to consider finally is the one occurring in the exponent of the integral representation for $E(x,t)$, and this function is $\sqrt{\phi^2(z,v)} = (v-z)\sqrt{F(z)}$. However, we cannot easily consider $(v-z)\sqrt{F(z)}$ as it has branch points in the half plane $\operatorname{Im} z \geq 0$. Thus we examine $\phi^2(z,v)$ itself and then we shall be able to make any necessary analytic continuations of $\sqrt{\phi^2(z,v)}$. The information we wish to obtain is: where is $\phi^2(z,v)$ real, and is it positive or negative there? In the following discussion we assume $v \neq 0, \pm 1$. By previous arguments at least one end of any contour on which $\text{Im}\, \phi^2(z,v) = 0$ must lie on the real axis. On the real axis $\phi^2(z,v)$ is real at ± 1, where it is negative; at $z = 0$, where it is positive; and at $z = v$ where $\phi^2(z,v)$ has a double zero. From the points $z = 0$ and $z = \pm 1$ there should be contours going into the upper half plane on which $\phi^2(z,v)$ is real. Leaving $z = v$ should be two contours on which $\phi^2(z,v)$ is real and on one $\phi^2(z,v)$ is positive, on the other $\phi^2(z,v)$ is negative. Of these two contours at most one can go to infinity. The remaining information that we have is that on the imaginary axis $\phi^2(z,v)$ is not real except at $z = z_o$ where $\phi^2(z,v)$ has a simple zero. Thus only one curve on which $\text{Im}\, \phi^2(z,v) = 0$ can cross the imaginary axis and the crossing is at $z = z_o$. We must now determine how to connect all the various curves. We make the connections for the cases of v small and v large and then conjecture a continuous transition between the limits. We use the properties that for v large $\phi^2(z,v)$ is approximately $v^2F(z)$ and for v small $\phi^2(z,v)$ is approximately $\psi(z)$.

In Figure 7a we draw the curves $\text{Im}\, \phi^2(z.v) = 0$ for v very large; in Figure 7b we draw the same figure showing how the curves approximate those of $\text{Im}\, v^2F(z) = 0$. To justify Figure 7 we note first that from (37) and (39) we may expand $\phi^2(z,v)$ near $z = 0$, and we obtain

$$\phi^2(z,v) = v^2|F(0)| + v^2 z\left(i|F'(0)| - \frac{2|F(0)|}{v}\right) + O(z^2).$$

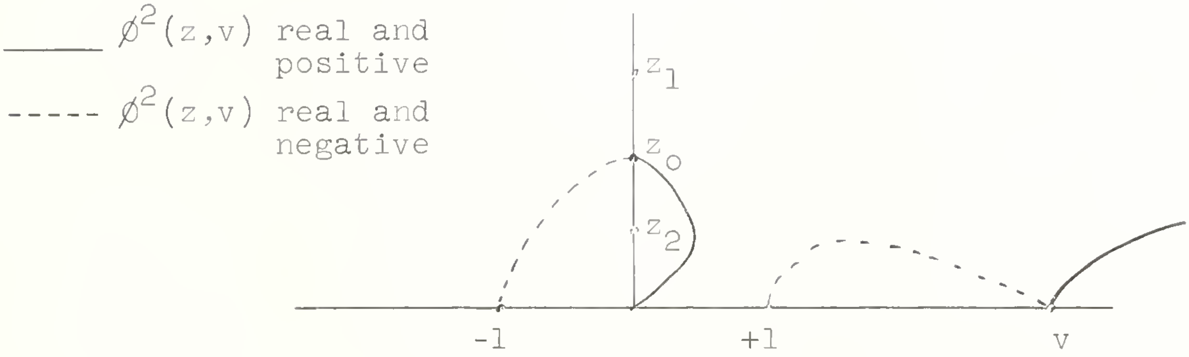

______ $\phi^2(z,v)$ real and positive
----- $\phi^2(z,v)$ real and negative
z_1
z_o
z_2
-1
+1
v

Figure 7a

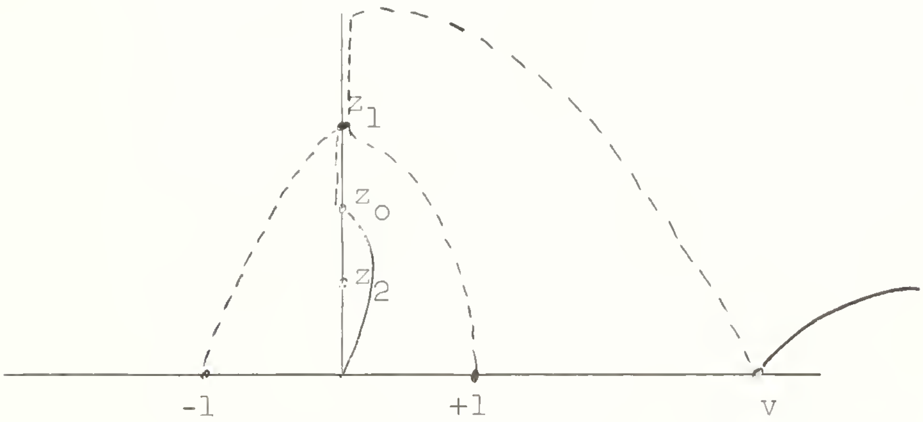
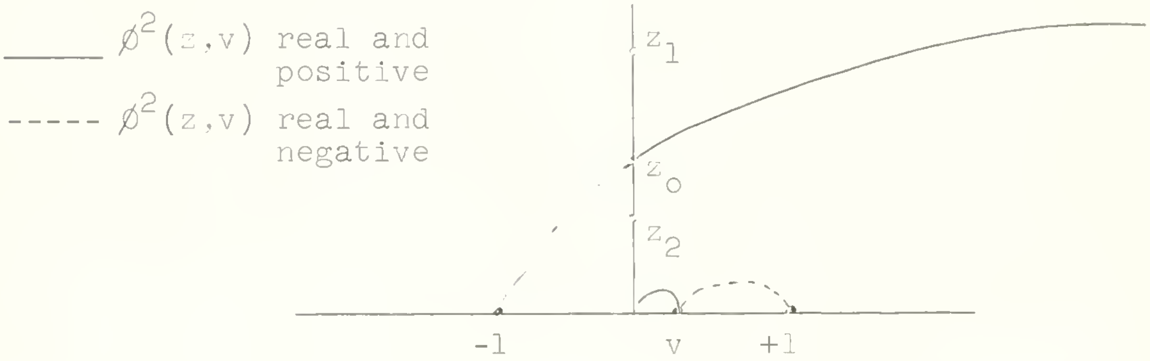


Figure 7b
Curves Im $\phi^2(z,v) = 0$ for v large

Thus, for $v > 0$ the curve on which $\phi^2(z,v)$ is real and positive leaves the origin and goes into the first quadrant, as shown. Next, let us consider the values of $\phi^2(z,v)$ on a semicircle of radius larger than 1 and $|z_1|$, but much less than v. If, for v large, $\phi^2(z,v)$ is never real and positive on that semicircle, then the contour line leaving the origin must go to z_o and then connect to -1. Since both contour lines from v cannot go to infinity, they must connect as shown. It remains to show that on our given semicircle $\phi^2(z,v)$ is not real and positive. Let us suppose the contrary, that there be infinitely many pairs v_i, z_i, with $v_i \rightarrow \infty$, such that $\phi^2(z_i,v_i)$ is real and positive. Since the semicircle is compact, we may pick a subset of z_j's which converge to some point ζ. As $(1 - \left(\frac{z_j}{v_j}\right)^2)F(z_j)$ is real and positive for all j, when $j \rightarrow \infty$, $z_j \rightarrow \zeta$ and $v_j \rightarrow \infty$, so that $F(\zeta)$ is real and positive, which is false. Again we have not shown that the contours from -1 and z_o do not meet at infinity. While we might show this with a little effort, since the result is not necessary below we ignore it.

In Figures 8a and 8b we give similar diagrams for $0 < v \ll 1$.



———— $\phi^2(z,v)$ real and positive
- - - - - $\phi^2(z,v)$ real and negative
z_1
z_o
z_2
-1
v
+1

Figure 8a

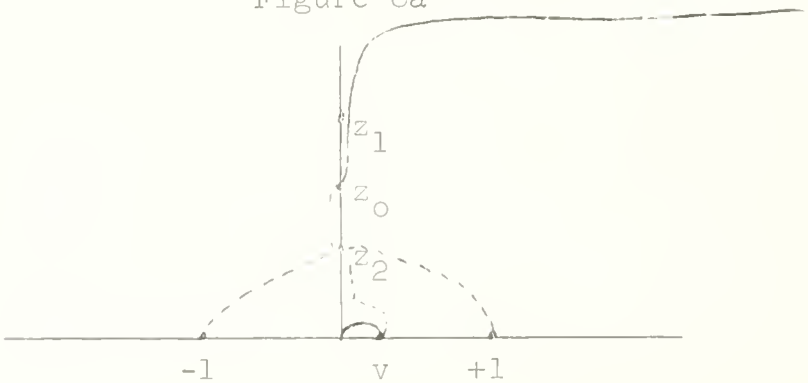


Figure 8b

Curves Im $\phi^2(z,v) = 0$ for v small

To justify Figures 8 let us consider first a semicircle with center at the origin and passing through z_2. By arguments similar to the one above, if $\phi^2(z,v)$ is real and positive on the semicircle as v tends to zero, then $\psi(z)$ is real and positive there -- a contradiction. Thus the curves from the origin and $z = v$ connect. By taking a similar semicircle with radius larger than 1 and $|z_1|$, we may show that for v small $\phi^2(z,v)$ is never real and negative there. Thus the remaining contours are correctly connected.

In order to make the transition between v small and v large we use the properties of $\chi(z)$ given in Figure 6. We recall that values of z on $\text{Im}\ \chi(z) = 0$ correspond to values of $v = \chi(z)$ such that $\frac{d}{dz}\phi^2(z,v) = 0$. In other words, each point on the curve $\text{Im}\ \chi(z) = 0$ is a double point of the contour lines of $\phi^2(z,v)$ for the appropriate value of v. From the general configuration of Figures 7 and 8, we see that only one curve on which $\phi^2(z,v)$ is real and negative appears in either quadrant; thus if any curves $\text{Im}\ \phi^2(z,v) = 0$ have a double point, then we expect that $\phi^2(z,v)$ is positive there. From figures 6, 7, and 8 we see that for v large the value of z for which $\frac{d}{dz}\phi^2(z,v) = 0$ is near z_1 while for v small it is near z_2. For no value of v should the double point lie inside the curve from $z = +1$ to $z = v$. As this curve corresponds to $\phi^2(z,v)$ real and negative we expect no double points on it, so that as $v \to 1$ we find that $z = v = +1$ is a double point of $(v-z)\sqrt{F(z)}$ -- a contradiction. Thus for v large the double point may be connected to the negative real axis by a curve not passing through any curves on which Im $\phi^2(z,v) = 0$. For small v no such connection is possible. Since all curves move continuously with respect to v we expect to find some value of v_c for which the double point crosses a curve Im $\phi^2(z,v_c) = 0$. Figure 9 illustrates this critical situation. In Figure 9 we have omitted the curve from $z = +1$ to $z = v$ on which $\phi^2(z,v)$ is real and negative.

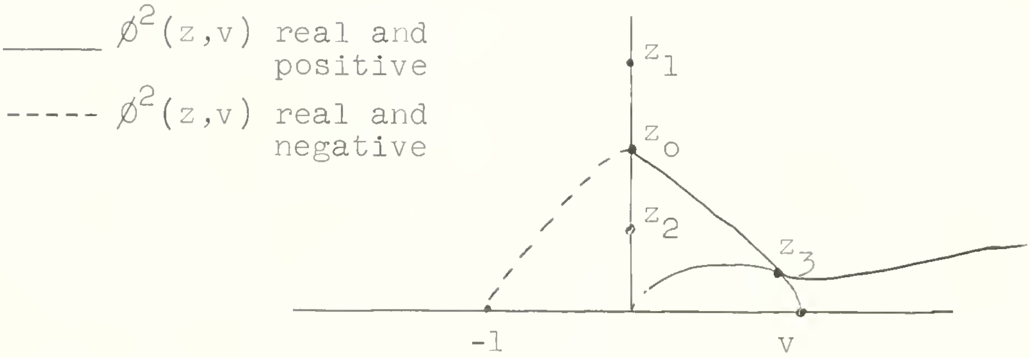

_____ $\phi^2(z,v)$ real and positive
----- $\phi^2(z,v)$ real and negative
z_1
z_o
z_2
z_3
-1
v

Figure 9
Curves Im $\phi^2(z,v) = 0$ for $v = v_c$

For $v < v_c$ we expect the stationary point of $\phi^2(z,v)$ to lie within the curve from -1 to z_o to infinity. For $v > v_c$ we expect the double point to be outside the curve from +1 to v to infinity. The value of v_c is given by the simultaneous solution of the relations

(42a) $$\text{Im } \chi(z_3) = 0$$

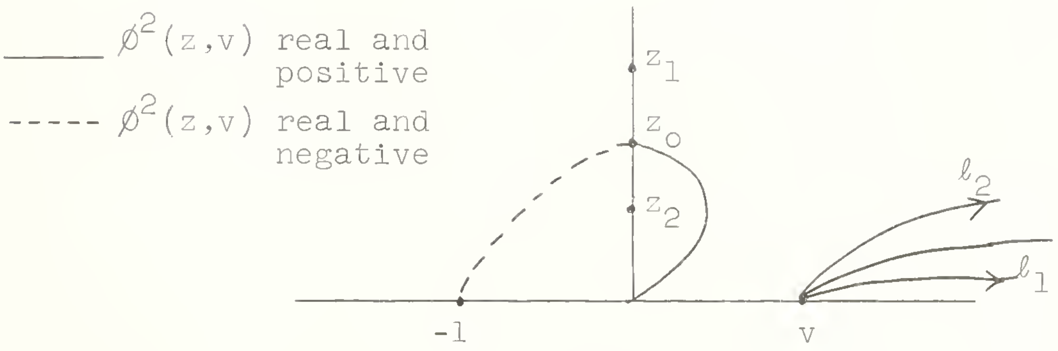
(42b) $$v_c = \chi(z_3)$$

(42c) $$\text{Im } \phi^2(z_3 . v_c) = 0$$

We show that the asymptotic expansions change character as v becomes larger than v_c. It is possible that several solutions of the equations (42) exist, but we assume this does not happen. The methods below would apply in any case.

We can now proceed to the principal task of obtaining asymptotic expansions of the electric field. Again we consider first the limit $x/t = v$ fixed and $t \to \infty$. We define a special branch of the square root such that $\operatorname{Im} \phi^2(z,v) > 0$. We call $\phi(z,v)$ the function $\phi^2(z,v)$ so defined. If we consider the analytic function $\phi(z,v)$ and perform an analytic continuation, then as we cross a curve on which $\phi^2(z,v)$ is real and positive $\phi(z,v)$ moves onto the other sheet of the Riemann surface on which $\operatorname{Im} \phi(z,v) < 0$.

For the case $v > v_c$ we define contours of integration in Figure 10. Again we have suppressed the curve from $z = v$ to $z = +1$ on which $\phi^2(z,v)$ is real and negative.



$\phi^2(z,v)$ real and positive
$\phi^2(z,v)$ real and negative
z_1
z_o
z_2
ℓ_2
ℓ_1
-1
v

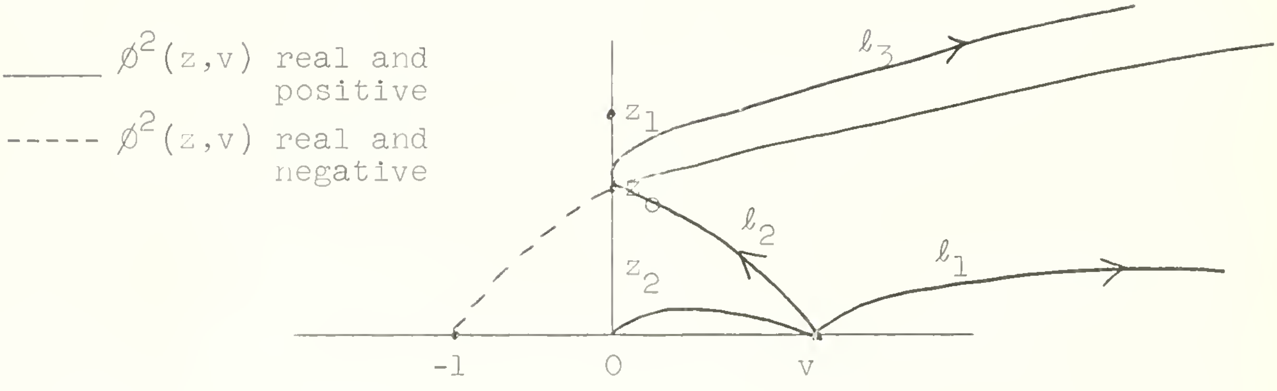
Figure 10
Definition of contours ℓ_1 and ℓ_2 when $v > v_c$

Starting from (18) for $E(vt,t)$ with ℓ_1 as the contour of integration, we may make an analytic continuation of $\phi(z,v)$ across the curve on which $\phi^2(z,v)$ is real and positive, and then we distort the contour of integration to ℓ_2 for one of the terms and we find

$$(43) \qquad E(vt,t) = \frac{\mathrm{Im}}{2\pi}\left\{ \int_{\substack{v \\ \text{on } \ell_1}}^{\infty} e^{it\phi(z,v)} G(z)dz + \int_{\substack{v \\ \text{on } \ell_2}}^{\infty} e^{-it\phi(z,v)} G(z)dz \right\}.$$

In equation (43) both terms are decreasing exponentials and we may derive equation (30).

Next we consider $0 < v = x/t < v_c$, v fixed and $t \rightarrow \infty$. In Figure 11 we define the relevant contours, again omitting the curve from v to $+1$.



_____ $\phi^2(z,v)$ real and positive
----- $\phi^2(z,v)$ real and negative
ℓ_3
z_1
z_0
ℓ_2
ℓ_1
z_2
-1
0
v

Figure 11
Definition of contours ℓ_1, ℓ_2, and ℓ_3 when $v < v_c$

We start from the same equation for $E(vt,t)$ with the contour of integration ℓ_1. On making the analytic continuation across the contour from z_o to infinity we may write

$$(44) \qquad E(vt,t) = \frac{\mathrm{Im}}{2\pi}\left\{ \int_{\substack{v \\ \text{on } \ell_1}}^{\infty} G(z)e^{it\phi(z,v)}dz + \int_{\substack{z_0 \\ \text{on } \ell_3}}^{\infty} e^{-it\phi(z,v)}dz\, G(z) \right\}$$

$$+ \frac{\mathrm{Im}}{2\pi}\int_{\substack{v \\ \text{on } \ell_2}}^{z_0} e^{-it\phi(z,v)}G(z)dz \ .$$

In equation (44) the first two integrals are decreasing functions of time and are bounded by some constant independent of x and t. However, the third integral is an increasing exponential. Thus we assert

$$(45) \qquad E(v,t) = \frac{\mathrm{Im}}{2\pi}\int_{\substack{v \\ \text{on } \ell_2}}^{z_0} e^{-it\phi(z,v)}G(z)dz + O(1).$$

From the earlier remarks we expect to find a value z_4 within the contour from -1 to z_0 to infinity at which $\left.\frac{d\phi}{dz}(z,v)\right|_{z=z_4} = 0$. Since $\phi(z,v) = 0$ at $z = z_0$ and $z = v$, and since $\mathrm{Im}\,\phi(z,v) > 0$ within the chosen contour we should be able to deform the contour of integration ℓ_2 so that it passes through z_4 and so that for large t the absolute value of the integrand is largest on ℓ_2 at z_4. We may then perform a saddle point asymptotic expansion and we get

$$(46) \qquad E(t,vt) \sim \frac{\mathrm{Re}}{2\pi}\left\{ G(z_4)e^{-it\sqrt{\phi(z_4v)}}\sqrt{\frac{\pi}{t\phi''(z_4,v)}} \right\} \ .$$

The square root in the exponent of (46) has negative imaginary part. the other square root is chosen with Re $\sqrt{\phi''(z_4,v)} > 0$. If $\phi''(z_4)$ is real and negative then (46) is ambiguous up to a sign which depends sensitively on the contour ℓ_2. As $v \rightarrow v_i$ from below, z_4 approaches z_3 and the growth exponent goes to zero.

Equation (46) includes the case x fixed and $t \rightarrow \infty$, with the replacements $z_4 \rightarrow z_2$, $v \rightarrow \frac{x}{t}$, and $\phi''(z_4,v) \rightarrow \psi''(z_2)$.

Qualitatively we see that the electric field is oscillating and exponentially growing behind a wavefront moving with speed $v = v_c$. Ahead of the front the disturbance disappears as $t \rightarrow \infty$. At a fixed point behind the front, as $t \rightarrow \infty$ the electric field only grows and does not oscillate. The maximum growth rate exponent behind the front is given by $-iz_2 \sqrt{F(z_2)}$.

We can say very little concerning a general unstable plasma as it is easy to produce various examples of plasmas for which the asymptotic expansion of the electric field has very irregular behavior. Perhaps the strongest statement one might make is that a plasma unstable in the sense that the dispersion relation has roots for Re $p > 0$, will almost assuredly be unstable in the sense that in some coordinate frame the electric field will grow exponentially in time. However, the coordinate frame need not be the rest frame (relative to the definition (4)). In fact, the frame may be moving with arbitrarily large speed relative to the rest frame. Also, the exponent of growth will usually be less than that predicted by the dispersion relation.

In the example considered above the region of instability was limited by the condition $|x|/t < v_c$. Again, systems exist for which there is no such v_c. With little effort the reader may verify that for appropriate values of the parameter δ the unperturbed distribution functions below have the properties just described.

$$(47a) \qquad \tilde{f}_o(u) = \frac{1}{\sqrt{\pi}} \frac{(u-a(\delta))^2 e^{-(u-b(\delta))^2}}{\delta^2 + \frac{1}{2}}$$

where

$$(47b) \qquad b = \frac{\delta}{\delta^2 + \frac{1}{2}}$$

$$(47c) \qquad a = \delta + b$$

and

$$(48) \qquad \tilde{f}_o(u) = \frac{2}{\sqrt{\pi}} \frac{1}{(1 + e^{-\delta^2})} e^{-u^2} \cos^2(\delta x).$$

VI. Three-Dimensional Green's Function

Having concluded our treatment of the one-dimensional case we now consider the three-dimensional case. We obtain the integral representations of the Green's function by an indirect procedure. Instead of determining functions satisfying the initial condition (2a), we exhibit a Green's function that satisfies the initial condition

$$(2a') \qquad \tilde{f}(\bar{r},\bar{v},0) = \frac{A}{r}\, g(\bar{v}) \ ,$$

where A is some constant. In some sense we should be able to obtain the desired Green's function by applying the operator $-\frac{1}{4\pi A}\ {}^{2}$ to that solution. In order to avoid the addition of an infinite amount of charge to the system, when we use (2a') we must impose (4). In the final Green's function we do not require (4).

It has been conjectured that if the equation (1a) included magnetic effects and if all of Maxwell's equations were added. then the only stable unperturbed distribution functions would be spherically symmetric in velocity space. Since we have not included magnetic forces we might expect our result to be relevant to real physical problems only when $f_o(\bar{v})$ is a function of $|\bar{v}|$. Thus, after obtaining the electric field, we consider the case in which $f_o(\bar{v})$ and $g(\bar{v})$ depend on $|\bar{v}|$ only, and we obtain asymptotic expansions as $t \longrightarrow \infty$.

We assert that the solution of the system of equations (1), (2a'), (2b), and (4) is

$$(49) \qquad \overset{\approx}{E}(\bar{r},t) = \mathrm{Im} \int d\Omega \; \hat{k} \int_{\frac{\hat{k}\cdot\bar{r}}{t}}^{\infty} dz \; G(z,\hat{k}) \cos\left(\sqrt{F(z,\hat{k})}\,(\hat{k}\cdot\bar{r}-zt)\right)$$

and

$$(50) \qquad \overset{\approx}{f}(\bar{r},\bar{v},t) = -\frac{e}{m}\,\mathrm{Im}\,\frac{\partial f_o(\bar{v})}{\partial v} \cdot$$

$$\left(\int d\Omega \; \hat{k} \int_{\frac{\hat{k}\cdot\bar{r}}{t}}^{\infty} \frac{dz \; G(z,\hat{k}) \sin\left(\sqrt{F(z,\hat{k})}\,(\hat{k}\cdot\bar{r}-zt)\right)}{(\hat{k}\cdot\bar{v}-z)\sqrt{F(z,\hat{k})}}\right) - \frac{2\pi^2 g(\bar{v})}{|\bar{r}-\bar{v}t|} \,.$$

In (49) and (50) $\hat{k}$ is a three dimensional unit vector and $d\Omega$ is the surface element on the k unit sphere. Formulas (49) and (50) are natural extensions of the one dimensional result and they may be derived from the transforms as in Section III. If we consider the integrand of the $\hat{k}$ integration in (49) or (50) we see that the expression is the same formula as (18) or (19). Thus the arguments of the one dimensional case show that (49) and (50) are well defined. Here we must use the condition (4) to guarantee that the integrals converge at infinity has no other function.

We next seek to show that $f(\bar{r},\bar{v},t)$ satisfies the initial condition (2a') and $\overset{\approx}{E}(\bar{r},t)$ the boundary condition (2b). If we define

$$(51) \qquad L(\hat{k},\bar{r},\bar{v},t) = \int_{\frac{\hat{k}\cdot\bar{r}}{t}}^{\infty} dz\, G(z,\hat{k})\, \frac{\sin((\hat{k}\cdot\bar{r}-zt)\sqrt{F(z,\hat{k})})}{(\hat{k}\cdot\bar{v} - z)\sqrt{F(z,\hat{k})}}$$

then

$$\tilde{f}(\bar{r},\bar{v},t) + \frac{2\pi^2 g(\bar{v})}{|\bar{r}-\bar{v}t|} = -\frac{e}{m}\, \mathrm{Im}\, \frac{\partial f_o(\bar{v})}{\partial\bar{v}} \cdot \int d\Omega\, \hat{k}\, L(\hat{k},\bar{r},\bar{v},t)\ .$$

For $\bar{r} = 0$ then $\lim_{t\to 0} L(\hat{k},\bar{r},\bar{v}t) = 0$, so that the initial condition is satisfied. For $|\bar{r}| \neq 0$, we divide the unit sphere into two regions, such that in the first $|(\hat{k}\cdot\bar{r})| > t^{1/2}$, and in the second $|(\hat{k}\cdot\bar{r})| \leq t^{1/2}$. In the first region we obtain easily $|L(\hat{k},\bar{r},\bar{v},t)| < ct^{1/2}$, and in the second region $|L(\hat{k},\bar{r},\bar{v},t)| < c'$. Since the total solid angle contained in the second region vanishes as $t \to 0$, we derive

$$\lim_{t\to 0} \left(\tilde{f}(\bar{r},\bar{v},t) + \frac{2\pi^2 g(\bar{v})}{|\bar{r}-\bar{v}t|}\right) = 0.$$

In order to derive the boundary conditions for $\tilde{\bar{E}}(\bar{r},t)$ we must modify the asymptotic expansions for $G(z,\hat{k})$ and $F(z,\hat{k})$. Since

$$G(z,\hat{k}) \sim \alpha_1/z^2\ ,$$

we can find positive constants α' and c such that

$$(52) \qquad |G(z,\hat{k})| < \frac{\alpha'}{|z|^2 + c}$$

in the whole half-plane Im $z \geq 0$. Similarly we have

$$(53) \qquad |F(z,\hat{k})| < \frac{\beta'}{|z|^2 + c'} \ .$$

Thus if $|\hat{k}\cdot\bar{r}/t| < |z|$ then $|\sqrt{F(z,\hat{k})}(\hat{k}\cdot r - zt)| < c''t$ for some constant c''. Hence

$$|\tilde{\bar{E}}(\bar{r},t)| < \int |d\Omega| \int_{\frac{\hat{k}\cdot\bar{r}}{t}}^{\infty} dz\, e^{(+c''t)} \frac{\alpha'}{|z|^2 + c} \ .$$

After some elementary integrals we derive

$$(54) \qquad |\tilde{\bar{E}}(\bar{r},t)| < c''' e^{c''t} \frac{t}{r} |\log(t/r)| \ ,$$

so that

$$(55) \qquad \lim_{\substack{|r| \to \infty \\ t>0,\text{fixed}}} |\tilde{\bar{E}}(\bar{r},t)| = 0 \ .$$

The proof that the first differential equation holds is the same as before. Again (1a) does not hold as it stands, but after the change of variables

$$\bar{r}' = \bar{r} - \bar{v}t$$

$$t' = t \ ,$$

(1a) becomes

$$(1a'') \qquad \frac{\partial \tilde{f}}{\partial t'}(\bar{r},\bar{v},t) + \frac{e}{m}\tilde{\bar{E}}(\bar{r},t)\cdot\frac{\partial f_0(\bar{v})}{\partial \bar{v}} = 0.$$

Equation (1a") is essentially trivial to verify. Equation (1c) is also trivially checked. We now consider (1b).

We derive readily

$$\overline{\nabla}\cdot\tilde{\overline{E}}(\overline{r},t) - 4\pi e \int \tilde{f}(\overline{r},\overline{v},t)d\overline{v} = -\frac{1}{t}\,\mathrm{Im}\int d\Omega\; G(z,\hat{k})\Big|_{z=\frac{\hat{k}\cdot\overline{r}}{t}}$$

$$+\; 8\pi^3 e \int d\overline{v}\; \frac{g(\overline{v})}{|\overline{r}-\overline{v}t|}\;,$$

so that we must show

$$\text{(56)}\qquad \mathrm{Im}\int d\Omega\; G(z,\hat{k})\Big|_{z=\hat{k}\cdot\overline{r}/t} = 8\pi^3 e \int d\overline{v}\; \frac{g(\overline{v})}{|\overline{v}-\overline{r}/t)|}\quad .$$

We can rewrite the defining equation for $G(z,\hat{k})$ as

$$G(z,\hat{k}) = 4\pi e \int d\overline{v}(\hat{k}\cdot \frac{\partial}{\partial\overline{v}}\, g(\overline{v}))\log(\hat{k}\cdot\overline{v}-z).$$

Hence we derive

$$-\mathrm{Im}\int d\Omega\; G(z,\hat{k})\Big|_{z=\hat{k}\cdot\overline{r}/t} = -2\pi^2 e \int d\Omega(d\overline{v})(\hat{k}\cdot \frac{\partial}{\partial\overline{v}}\, g(\overline{v}))\mathrm{sgn}(\hat{k}\cdot(\overline{v}-\overline{r}/t))$$

The $\hat{k}$ integration is elementary and (56) follows after an integration by parts.

While we have obtained the Green's function for the equations with the initial condition (2a'). we have not yet solved the original problem with initial condition (2a). We assert that

(57a) $$\overline{E}(\overline{r},t) = \frac{1}{8\pi^3} \nabla_r^2 \tilde{\overline{E}}(\overline{r},t)$$

and

(57b) $$f(\overline{r},\overline{v},t) = \frac{1}{8\pi^3} \nabla_r^2 \tilde{f}(\overline{r},\overline{v},t)$$

are the desired solutions. Since the function $\tilde{f}(\overline{r},\overline{v},t)$ is not twice differentiable with respect to $\overline{r}$, $\nabla^2\tilde{f}$ must be a distribution.(3) To see the meaning of a distribution as used here we consider functions $\phi(\overline{r})$ which are two times continuously differentiable and which vanish at infinity at least as fast as $1/r^4$, and whose first two derivatives also vanish that fast. To define the distribution $f(\overline{r},\overline{v},t)$ we must define all integrals $\int \phi(\overline{r})f(\overline{r},\overline{v},t)d\overline{r}$, but we have naturally that $\int(d\overline{r})\phi(\overline{r})f(\overline{r},\overline{v},t) \equiv \frac{1}{8\pi^3} \int \nabla^2\phi(\overline{r})\tilde{f}(\overline{r},\overline{v},t)d\overline{r}$. Thus for an arbitrary $\phi(\overline{r})$ of the class considered, the functions

$$\overline{\mathcal{E}}(\overline{r},t) = \frac{1}{8\pi^3} \int \nabla^2\phi(r') \tilde{\overline{E}}(\overline{r}-\overline{r}',t)(d\overline{r}')$$

$$\mathcal{F}(\overline{r},\overline{v},t) = \frac{1}{8\pi^3} \int \nabla^2\phi(r') \tilde{f}(\overline{r}-\overline{r}',\overline{v},t)(dr')$$

satisfy the differential equations and $\overline{\mathcal{E}}(\overline{r},t)$ vanishes, $|\overline{r}| \rightarrow \infty$, and $\mathcal{F}(\overline{r},\overline{v},0) = -\frac{1}{4\pi} \int \nabla^2\phi(\overline{r}') \frac{1}{|\overline{r}'-\overline{r}|} (d\overline{r}') = \int \phi(\overline{r}')\delta(\overline{r}-\overline{r}')(d\overline{r}')$. Hence, in the sense of distributions, or weak solutions, $\overline{E}(\overline{r},t)$ and $f(\overline{r},\overline{v},t)$ are the desired solutions.

The electric field in the Green's function is actually a proper function, for we derive easily

$$\overline{E}(\overline{r},t) = -\frac{\text{Im}}{(2\pi)^3}\int d\Omega\,\hat{k}\int_{\frac{\hat{k}\cdot\overline{r}}{t}}^{\infty} dz\; G(z,\hat{k})F(z,\hat{k})\cos\left(\sqrt{F(z,\hat{k})}(\hat{k}\cdot\overline{r}-zt)\right)$$

$$-\frac{\text{Im}}{8\pi^3 t}\frac{\partial}{\partial\overline{r}}\cdot\int d\Omega\;\hat{k}\,\hat{k}\,G\left(\frac{\hat{k}\cdot\overline{r}}{t},\,\hat{k}\right),$$

and after some elementary integrals we obtain

$$\overline{E}(\overline{r},t) = \frac{e}{t^2}\int (d\overline{v})g(\overline{v})\,\frac{\partial}{\partial v}\,\frac{1}{|\overline{v}-\overline{r}/t|} \tag{58}$$

$$-\frac{\text{Im}}{(2\pi)^3}\int d\Omega\;\hat{k}\int_{\frac{\hat{k}\cdot r}{t}}^{\infty} dz\; G(z,\hat{k})F(z,\hat{k})\cos\left(\sqrt{F(z,\hat{k})}(\hat{k}\cdot\overline{r}-zt)\right)$$

We might obtain an almost comparable formula for the distribution function, but it is very cumbersome and of little interest. However we note that the contour integral in (58) now converges at infinity even though we no longer require that (4) hold. We might easily show that the same is true of the distribution function, for if in (50) we take the path of integration from $\hat{k}\cdot\overline{r}/t$ to ω, Im $\omega > 0$, and from ω to infinity, then we could apply the Laplacian to the second integral and the resultant integral converges at infinity without assuming (4). We could then show that those forms satisfy the various equations. We omit the proofs as there is nothing new in them. In brief, (57) and (58), appropriately interpreted, are the solution to our original system of equations without (4).

When $f_o(\overline{v})$ and $g(\overline{v})$ are spherically symmetric $F(z,\hat{k})$

and $G(z,\hat{k})$ are independent of $\hat{k}$, so that

$$(59)\qquad \bar{E}(\bar{r},t) = -\frac{4\pi e\hat{r}}{rt}\int_{r/t}^{\infty} vg(v)dv$$

$$+\frac{\operatorname{Im}\hat{r}}{4\pi^2}\left\{\int_{-r/t}^{\infty} dz\; G(z)\left(\sqrt{F(z)}\,\frac{\sin((r+zt)\sqrt{F(z)})}{r}+\frac{\cos(\sqrt{F(z)}(r+zt))}{r^2}\right)\right.$$

$$\left.-\int_{r/t}^{\infty} dz\; G(z)\left(\sqrt{F(z)}\,\frac{\sin((r-zt)\sqrt{F(z)})}{r}+\frac{\cos(\sqrt{F(z)}(r-zt))}{r^2}\right)\right\}$$

In (59) in the last two contour integrals the contours must be the same near infinity so that the integral of the sum of the integrands may converge.

As is well known(4), any spherically symmetric distribution function is stable. Thus we may use the methods of Section IV to obtain asymptotic expansions of $\bar{E}(\bar{r},t)$. The expansion in the limit $r/t = v$ fixed, $t \to \infty$, is very complicated and none too illuminating, but with little effort one may show that

$$(60)\qquad \bar{E}(\bar{r},t) \sim \frac{\hat{r}}{t^4}\,h(v).$$

In the limit $t \to \infty$, r fixed, we obtain

$$(61)\qquad \bar{E}(\bar{r},t) = \frac{\alpha_o\omega_p^3\,\hat{r}\,r^2}{6\pi\sqrt{\pi}}\;\frac{\cos(\omega_p t - \pi/4)}{(\omega_p t\beta_2)^{3/2}} + O(1/t^2)\ .$$

VII. Two Dimensional Green's Function

This section duplicates much of the material of the last section for the two dimensional case. From transform techniques we are led to consider the expressions

$$(62)\quad \overline{E}(\overline{r},t) = -\frac{Re}{2\pi^2}\int d\Omega\,\hat{k}\int_{\frac{\hat{k}\cdot r}{t}}^{\infty} dz\; G(z,\hat{k})\sqrt{F(z,\hat{k})}\,\sin\!\left(\sqrt{F(z,\hat{k})}\,(\hat{k}\cdot\overline{r}-zt)\right)$$

$$+\frac{1}{2\pi^2 t}\,Re\int d\Omega\;\hat{k}\;G(z,\hat{k})\Big|_{\hat{k}\cdot\overline{r}/t}$$

and

$$(63')\quad f(\overline{r},\overline{v},t) =$$

$$-\frac{e}{2\pi^2 m}\,Re\left\{\frac{\partial f_o(\overline{v})}{\partial\overline{v}}\cdot\int d\Omega\;\hat{k}\int_{\frac{\hat{k}\cdot\overline{r}}{t}}^{\infty} dz\; G(z,\hat{k})\;\frac{\cos\!\left(\sqrt{F(z,\hat{k})}\,(\hat{k}\cdot\overline{r}-zt)\right)}{(\hat{k}\cdot v-z)}\right\}$$

$$+\;g(\overline{v})\,\delta(\overline{r}-vt).$$

Now all vectors have two components and $d\Omega$ is the "surface element" on the "two-sphere", i.e., $d\Omega = d\theta$. In the process of verifying one of the differential equations we obtain another form for the unusual integral appearing in (62). Since the integrand in (63') is not properly defined, we consider an alternate form which is meaningful:

$$(63)\quad f(\bar{r},\bar{v},t) = -\frac{e}{2\pi^2 m}\,\mathrm{Re}\int \frac{\partial f_o(\bar{v})}{\partial \bar{v}}\cdot\hat{k}\,d\Omega \int_{\frac{\hat{k}\cdot\bar{r}}{t}}^{\infty} dz\,\log(\hat{k}\cdot\bar{v}-z)\times$$

$$\frac{d}{dz}\left(G(z,\hat{k})\cos\left(\sqrt{F(z,\hat{k})}\,(\hat{k}\cdot\bar{r}-zt)\right)\right)$$

$$-\frac{e}{2\pi^2 m}\,\mathrm{Re}\int d\Omega\;\hat{k}\cdot\frac{\partial f_o(\bar{v})}{\partial\bar{v}}\;G\!\left(\frac{\hat{k}\cdot\bar{r}}{t},\,\hat{k}\right)\log\!\left(\hat{k}\cdot\bar{v}-\frac{\hat{k}\cdot\bar{r}}{t}\right)$$

$$+\;g(\bar{v})\,\delta(\bar{r}-\bar{v}t).$$

By the exact parallels of the arguments presented previously we might easily show that the Green's function is well-defined, that it satisfies the initial condition (2a) and the boundary condition (2b). Clearly, $\bar{E}(\bar{r},t)$ is curl free, so that (1c) holds, and after the usual transformation of variables, $\bar{r}' = \bar{r}-\bar{v}t$, and $t' = t$, equation (1a) also follows.

If we set

$$(64)\quad \bar{\mathcal{E}}(\bar{r},t) = \frac{1}{2\pi^2 t}\,\mathrm{Re}\int d\Omega\;\hat{k}\;G\!\left(\frac{\hat{k}\cdot\bar{r}}{t},\,\hat{k}\right)\;,$$

then we derive easily that

$$(65)\quad \bar{\nabla}\cdot\bar{E}(\bar{r},t) - 4\pi e\int f(\bar{r},\bar{v},t)(d\bar{v}) = \bar{\nabla}\cdot\,\bar{\mathcal{E}}(\bar{r},t) + \frac{4\pi e}{t^2}\,g\!\left(\frac{\bar{v}}{t}\right)\;.$$

From the modified form of the representation of $G(z,k)$, we learn

$$(66)\quad \bar{\mathcal{E}}(\bar{r},t) = \frac{2e}{\pi t}\,\mathrm{Re}\int d\Omega\;\hat{k}\left(\hat{k}\cdot\,\frac{\partial g(\bar{v})}{\partial\bar{v}}\right)\log\!\left(\hat{k}\cdot\left(\bar{v}-\frac{\bar{r}}{t}\right)\right)(d\bar{v})\;.$$

With the help of standard integrals we obtain

$$\mathrm{Re} \int d\Omega\, \hat{k}\hat{k} \log \hat{k}\cdot a = \pi \hat{a}\hat{a} + c\mathrm{II} ,$$

where II is the unit dyad and c is some constant. Hence (66) becomes

$$(67) \qquad \bar{\mathcal{E}}(\bar{r},t) = 2e \frac{\partial}{\partial \bar{r}} \int (d\bar{v}) g(\bar{v}) \log(\bar{v} - \frac{\bar{r}}{t}).$$

From (67) and (65) we see that (1b) is satisfied.

Since there does not seem to be a simple form for the electric field when $g(\bar{v})$ and $f_o(\bar{v})$ are functions of $|v|$ only, it is difficult to obtain both of the asymptotic expansions obtained previously. However, in the cylindrically symmetric case -- for which the distribution function is again stable(4) -- it is possible to obtain an asymptotic expansion in the limit r fixed, $t \longrightarrow \infty$. In this case the one dimensional methods may be applied to the contour integral and then the $\hat{k}$ integral may be performed. Following this technique we obtain

$$(68) \qquad \bar{E}(\bar{r},t) = -\pi\omega_p^2 \alpha_o \hat{r}\, r \frac{\sin \omega_p t}{\omega_p - \beta_2} + O(\frac{1}{t^2}) .$$

Footnotes

(1) For various approaches to this problem see, for instance,

L. Landau, J. Phys. (USSR) 10, 25 (1940),

N. G. Van Kampen, Physica 21, 949 (1955),

and the comprehensive study of Backus,

G. Backus, J. Math. Phys. 1, 178 (1960).

(2) A preliminary account of this work was presented at the November 1961 meeting of the Division of Plasma Physics of the American Physical Society at Colorado Springs, Colorado.

(3) See Courant-Hilbert, Methods of Mathematical Physics, Vol. II Interscience-Wiley (1962), to appear shortly.

(4) See, for instance, G. Backus, op. cit.

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